Controllable Quantum Switchboard

D. Kaszlikowski,[1] L. C. Kwek,[2] C. H. Lai,[1] and V. Vedral[3]

[1]*Department of Physics, National University of Singapore, 2 Science Drive 3, Singapore 117542*
[2]*Nanyang Technological University, National Institute of Education, 1, Nanyang Walk, Singapore 637616*
[3]*The School of Physics and Astronomy, University of Leeds, Leeds, LS2 9JT, United Kingdom*

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All quantum information processes inevitably requires the explicit state preparation of an entangled state. Here we present the construction of a quantum switchboard which can act both as an optimal quantum cloning machine and a quantum demultiplexer based on the preparation of a four-qubit state.

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Many quantum information processes require the explicit preparation of specially entangled quantum states. Two-qubit maximally entangled state often called Bell state, for instance, form an essential quantum resource needed in quantum teleportation [1]. The preparation of three-qubit maximally entangled state (like GHZ) could be harnessed for secure secret sharing [2]. In one-way quantum computing, a four-qubit entangled state called cluster state provides an efficient implementation of a universal quantum gate: arbitrary single-qubit unitary operation and the CNOT gate [3].

It is interesting to note that entangled states which are used as a common resource in quantum information processes generally need not even be maximally entangled at all. As long as the state is genuinely entangled, quantum computation and communication will generally be better than the classical counterparts. In particular, the non-maximally entangled W state has been experimentally implemented and proposed for controlled quantum teleportation and secure communication [4].

An essential component of any quantum computation is the ability to spread quantum information over various parts of quantum computer. The parts then undergo separate evolutions depending on the type of the quantum information processing we wish to implement. Ultimately we need be capable of navigating the relevant part of the information into a designated output. In a classical computer this flow of information is achieved through a controllable switch. Is it possible to design a quantum analogue for such a device? An added complexity in a quantum switch would be the requirement that the information flows down many possible channels coherently as well as the possibility of channeling it in one selected direction.

Ideally we would like to realize the simplest such a device with the least number of qubits needed for this purpose. In addition these qubits will in practice be implemented in a physical system which will determine the nature of the qubits and couplings between them. Therefore, when designing our switch we should also take into account realistic interaction between the qubits, which severely limits the number of possible Hamiltonians to execute such a quantum switch. Here we present a possible implementation of the switch that fulfills of all the above requirements.

Let us consider an interesting four-qubit state described by

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(|(11)_{12}\rangle|(11)_{34}\rangle - |(11)_{14}\rangle|(11)_{23}\rangle\right), \qquad (1)$$

where $|(11)_{ij}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_i|1\rangle_j - |1\rangle_i|0\rangle_j)$ is the singlet state. Throughout the paper we use the following notation for the Bell states $|(ab)\rangle = \sum_{k=0}^{3}\frac{(-1)^{kb}}{\sqrt{2}}|k,k+a\rangle$ with summation modulo 2. It turns out that this state is ideally suited for a quantum switchboard, i.e., a circuit that can be used to direct the flow of quantum information in a controllable manner. An interesting property of the presented switchboard is that in the case of failure the information is not entirely lost.

The first qubit of the state (1) belongs to Alice, the second one to Bob, the third one to Charlene and the last one to Dick. Suppose Alice attaches some auxiliary qubit to the first qubit and perform a joint Bell measurement. Immediately after getting one of the four possible outcomes, she broadcasts two (classical) bits of information to Bob and Charlene as it is in the usual teleportation scheme. At this point, it is not necessary for Dick to know these two bits of information.

Bob and Charlene can recover the state of the auxiliary qubit with the fidelity $\frac{5}{6}$ by applying appropriate unitary transformation based on the knowledge of the broadcast classical bits. The given state at the beginning does not provide a universal cloning machine for three copies of the cloned state [5]. Thus, the qubit belonging to Dick is related to the Alice's auxiliary qubit with the "classical" fidelity $\frac{1}{3}$, i.e., the fidelity that can be achieved without prior entanglement. It is interesting to note that Bob and Charlene possess the optimum fidelity achievable under a symmetric cloning machine. Dick's fidelity is allowed since there is no limitation on the production of clones with the fidelity below $\frac{2}{3}$.

Thus the presented protocol behaves like an optimal telecloner [6]. However, there is still an unused qubit held by Dick. Depending on Alice's decision regarding to whom she wishes ultimately to send her auxiliary qubit, say Bob (Charlene) for instance, she can direct Dick to send his qubit to Charlene (Bob). As soon as Charlene receives Dick's qubit, he can perform a Bell measurement on his qubit with Dick's qubit and send the results of his

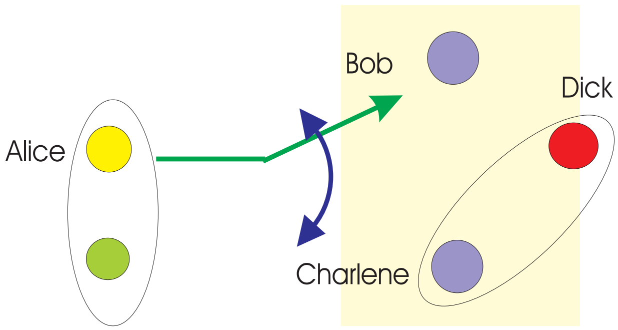

FIG. 1: Suppose Alice wishes to send her auxiliary qubit to Bob. She can direct Dick to send his qubit to Charlene. Charlene then performs a Bell measurement on his qubit with Dick's qubit and send the results of his measurement to Bob. Using the information from Charlene, Bob can perfectly recover the state of the Alice's auxiliary qubit.

measurement to Bob. Using the information from Charlene, Bob can perfectly recover the state of the Alice's auxiliary qubit.

The situation is entirely symmetric, i.e., Dick can send his qubit to Bob instead of Charlene with the result that now Charlene can obtain Alice's auxiliary qubit with perfect fidelity. In short the state acts as a quantum switchboard in which Alice can direct optimal clones to Bob and Charlene or perform perfect quantum teleportation to Bob or Charlene by utilizing Dick's qubit as in a quantum demultiplexer. A schematic diagram of this quantum switchboard protocol is shown in Fig. 1. By directing Dick's qubit to either Bob (or Charlene), Alice can effectively transfer the unknown auxiliary qubit to Charlene (or Bob). Moreover, she can delay the transfer process to a later time as long as she has effective control over Dick's qubit.

Incidentally, other shared states may be able to achieve some aspect of our quantum switch but it is difficult to find a state with all desired properties. For instance with the GHZ state, shared among Alice, Bob and Charlene, one could in principle provide perfect quantum teleportation to both Bob and Charlene, but without the additional benefit of an optimal quantum cloner. In this case, Alice teleclones to both Bob and Charlene with a classical fidelity of 2/3. The eventual quantum teleportation to Bob (or Charlene) is performed with a measurement in the basis $1/\sqrt{2}(|0\rangle \pm |1\rangle)$.

It is also interesting to note that the sheer presence of singlets or dimer-like bonds in the four-qubit state may make it more robust to certain types of noise. One example would be fluctuating magnetic field or polarization drift, depending on how we implement our qubits. This kind of fault tolerance is absent in the GHZ state.

Let us now prove the above statements. It is convenient to write the state $|\psi\rangle$ in the following way

$$|\psi\rangle = \frac{1}{2\sqrt{3}}(3|(11)_{12}\rangle|(11)_{34}\rangle + |(10)_{12}\rangle|(10)_{34}\rangle + |(01)_{12}\rangle|(01)_{34}\rangle - |(00)_{12}\rangle|(00)_{34}\rangle). \quad (2)$$

We can immediately see that the state shared by Alice and Bob is the Werner state with $\frac{1}{3}$ of noise. Taking into account that the fidelity of teleportation for the Werner state with the noise fraction $1-p$ is given by $\frac{p+1}{2}$ [7] we see that the fidelity of Bob's qubit is $\frac{5}{6}$. It can be checked that the state between Alice and Dick is the Werner state that is an equal mixture of the three Bell states $|(10)\rangle, |(01)\rangle, |(00)\rangle$. Thus Dick's clone of Alice's auxiliary qubit has the fidelity $\frac{1}{3}$, which is the fidelity achievable classically.

The state $|\psi\rangle$ is symmetric with respect to Bob and Charlene

$$|\psi\rangle = \frac{1}{2\sqrt{3}}(3|(11)_{13}\rangle|(11)_{24}\rangle + |(10)_{13}\rangle|(10)_{24}\rangle + |(01)_{13}\rangle|(01)_{24}\rangle - |(00)_{13}\rangle|(00)_{24}\rangle). \quad (3)$$

therefore Charlene's clone has the same fidelity as Bob's one.

Let us now write the state $|\psi\rangle$ together with the Alice's auxiliary qubit $|\alpha\rangle$ (particle with 0 index) in the form suitable for further analysis. To focus our attention we consider the scenario where Dick sends his qubit to Charlene. We have

$$|\alpha\rangle|\psi\rangle = \frac{1}{4\sqrt{3}} \sum_{k,l,m,n=0}^{1} \lambda_{kl} |(mn)_{01}\rangle \otimes \otimes U_{mn,kl}|\alpha\rangle|(kl)_{34}\rangle, \quad (4)$$

where $\lambda_{11} = 3, \lambda_{01} = \lambda_{10} = 1, \lambda_{00} = -1$ and $U_{mn,kl}$ is a usual unitary transformation that appears in the process of teleportation with the Bell state $|(kl)\rangle$ and with the outcome of Bell measurement (mn). For instance, $U_{01,11} = \sigma_x$.

Suppose now that the outcome of Alice's measurement is (mn). The collapsed state $|\chi_{mn}\rangle$ shared by Bob, Charlene and Dick is

$$|\chi_{mn}\rangle = \sum_{k,l=0}^{1} \frac{\lambda_{kl}}{2\sqrt{3}} U_{mn,kl}|\alpha\rangle|(kl)_{34}\rangle. \quad (5)$$

Therefore, Bob's state ρ_{mn} reads

$$\rho_{mn} = \frac{1}{12} \sum_{k,l=0}^{1} |\lambda_{kl}|^2 U_{mn,kl}|\alpha\rangle\langle\alpha|U^\dagger_{mn,kl}. \quad (6)$$

After receiving two bits (mn) of classical information from Alice, Bob can recover Alice's state with the fidelity $\frac{5}{6}$ as mentioned before. However, when Charlene performs the Bell measurement on his and Dick's qubit, obtains the result (kl) and sends it to Bob, Bob receives the state

$$\frac{\lambda_{kl}}{|\lambda_{kl}|} U_{mn,kl}|\alpha\rangle, \quad (7)$$

which he can transform back to the state $|\alpha\rangle$ by applying the inverse unitary transformation $U^{\dagger}_{mn,kl}$.

The symmetry of the state $|\psi\rangle$ allows us to repeat the same argument for the case in which Alice decides to send her qubit to Bob so that now Charlene can obtain the state $|\alpha\rangle$ with perfect fidelity. It is interesting to note that the relative phase between the components of the state $|\psi\rangle$ is crucial for desired functionality. Other phase choices or, for that matter, the complete lack of coherence, will not give us the same quantum switch.

Finally we emphasise that our quantum switch state is a ground state, albeit degenerate, of the Majundar-Ghosh (MG) model [10]. This spin chain belongs to a class of many- body Hamiltonians that provide a good qualitative account of materials like $Cu_2(C_5H_{12}N_2)_2Cl_4$, $CuGeO_3$ and $YCuO_{2.5}$[11]. Therefore our quantum switch is very realistic since it may already exist in some solid state systems. MG is essentially a one-dimensional quantum spin chain with nearest- and next-nearest-neighbor exchange interactions described by the the Hamiltonian

$$H_{MG} = J\sum_{i=1}^{N}\left(2\vec{S}_i\vec{S}_{i+1} + \alpha\vec{S}_i\vec{S}_{i+2}\right), \tag{8}$$

where $J > 0$ and N is the number of sites in the one-dimensional lattice with periodic boundary condition. The Hamiltonian is exactly solvable for $\alpha = 1$ and has a quantum phase transition from an ordered phase to a disordered spin-liquid-like phase as α varies from zero to some critical value $\alpha_{crit} = 0.482$[12].

At $\alpha = 1$ and for an even N, there is a two-fold degenerate ground state subspace spanned by two dimer configurations

$$\begin{array}{l} |(11)_{12}\rangle|(11)_{34}\rangle\ldots|(11)_{(N-1)N}\rangle \\ |(11)_{23}\rangle|(11)_{45}\rangle\ldots|(11)_{N1}\rangle, \end{array} \tag{9}$$

superposition of which, for $N = 4$, gives us the state $|\psi\rangle$.

In conclusion, we have provided a quantum switchboard which could act both as an optimal quantum cloning machine or a quantum demultiplexer. Moreover, we also note that it is possible to extend the switchboard to higher spins and higher dimensional spaces as long as we have a configuration of dimer-like neighboring bonds. We also note that apart from spin chains, it is possible that the four-qubit state considered in this paper could also be created from multi-photon entangled states generated with spontaneous parametric down conversion and linear optics apparatus.

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